Math 105 Chapter 3:

In real life applications of functions, a function represents some quantity that we want to study how it changes. A big question we ask is when is the quantity at a maximum/minimum and if so how do we find it. The process of finding the maximum/minimum of a function is called optimization.

Definition: • A function f is at a local maximum at (a,b) if

$$f(x,y) \leq f(a,b)$$

For all (x,y) around a small ball around (a,b).

• A function f is at a local minimum at (a,b) if

$$f(x,y) \geq f(a,b)$$

For all (x,y) around a small ball around (a,b).

• Local maxima/minima are also called local extrema.

Recall in calculus 1, if $f(x)$ had a local extrema, then $f'(x)=0$. We have an analogous result for multivariate functions.

Theorem: If f has a local extrema at (a,b) then,

$$\nabla f(a,b) = 0$$

or, $$f_x(a,b) = f_y(a,b) = 0$$

Definition: A point (a,b) in the interior of $D(f)$ (Domain of f.) is a critical point of f if either $\nabla f(a,b)$ is zero or doesn't exist.

Eg Find the critical points of $f(x,y) = xye^{x-y}$

$$f_x(x,y) = ye^{x-y} + xye^{x-y} = e^{x-y}y(1+x)$$

$$f_y(x,y) = xe^{x-y} - xye^{x-y} = e^{x-y}x(1-y)$$

$$\left.\begin{array}{l} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{array}\right\} \Rightarrow \begin{array}{ll} y(1+x) = 0 & (1) \\ x(1-y) = 0 & (2) \end{array}$$

If $y=0$ then (2) implies

$$0 = x(1-y) = x$$

If $y \neq 0$ then (1) implies $1+x=0$, or $x=-1$. Since $x \neq 0$, we must have that $1-y=0$ by (2) so $y=1$.

Therefore the 2 critical points are $(0,0)$, $(-1,1)$.

Now similar calculus 1, just because we have the derivative is zero, it does <u>not</u> imply that we have a local max/min (eg $y=x^3$), the same annoyance exists for multivariate functions, illustrated by the example below:

eg a) $f(x,y) = x^2+y^2$

b) $g(x,y) = -x^2-y^2$

c) $h(x,y) = -x^2+y^2$

a) $f_x = 2x$, $f_y = 2y$, so:

$$Df(x,y) = 0 \Rightarrow x = y = 0.$$

Thus $(0,0)$ is the only critical point.

Notice that for all $(x,y) \in \mathbb{R}^2$, we have

$$f(x,y) = x^2+y^2 \geq 0 = f(0,0)$$

So $(0,0)$ is a <u>local maximum.</u>

b) $y_x = -2x$, $g_y = -2y$, so:

$$\nabla y(x,y) = 0 \Rightarrow x = y = 0$$

Notice that for all $(x,y) \in \mathbb{R}^2$, we have

$$g(x,y) = -x^2-y^2 \leq 0 = y(0,0)$$

So $(0,0)$ is a <u>local minimum</u>

c) $h_x = -2x$, $h_y = 2y$, so:

$$\nabla h(x,y) = 0 \Rightarrow x = y = 0$$

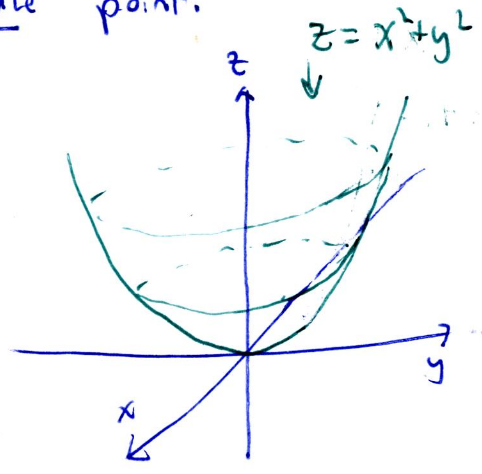
But for all $x \in \mathbb{R}$,

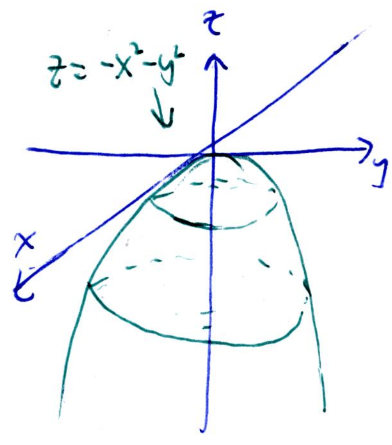
$$h(x,0) = -x^2 \leq 0 = h(0,0)$$

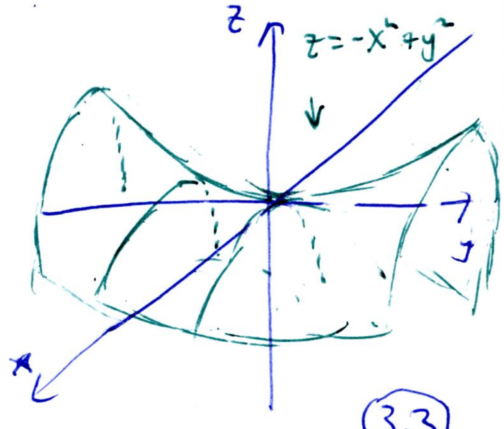
And for all $y \in \mathbb{R}$,

$$h(0,y) = y^2 \geq 0 = h(0,0)$$

So $(0,0)$ is <u>neither</u> a local maximum/maximum but rather a <u>saddle</u> point.





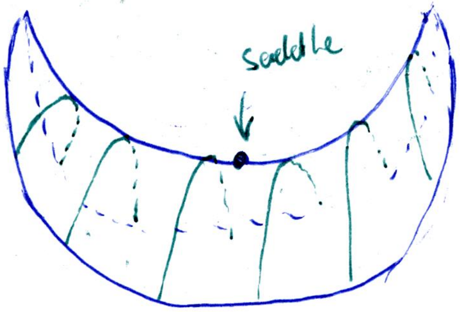


3.3

Definition: A point (a,b) is called a saddle point critical point but in every ball around (a,b) there some points (x_1,y_1), (x_2,y_2) such that

$$f(x_1,y_1) > f(a,b)$$
$$f(x_2,y_2) < f(a,b)$$

Think saddle or pringle chip.



How do determin if a critical point is a local min/max or saddle? Well in calc I we used the second derivative test. But we have 4 different second derivatives, so what do we do? Turns out there is a generalization that uses all 4 second derivatives. YAY!

Definition! Given a function $f(x,y)$ the Hessian is

$$D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - f_{xy}(x,y)\, f_{yx}(x,y)$$

Theorem! (Second derivative test) Let (a,b) be a critical point of f such that $Df(a,b)$ exists, and the second order partials exist at (a,b) and are continuous on some disc around (a,b). Then:

① If $D(a,b) < 0$, then (a,b) is a saddle point of f.

② If $D(a,b) > 0$, $f_{xx}(a,b) > 0$, then (a,b) is a local min.

③ If $D(a,b) > 0$, $f_{xx}(a,b) < 0$, then (a,b) is a local max.

Note that if $D(a,b)=0$ the test is inconclusive.

if $D(a,b)>0$ but $f_{xx}(a,b)=0$ then the test is inconclusive.

Eg. Classify all critical points of.

$$f(x,y)=x^3+x^2y-y^2-4y$$

Solution: $f_x = 3x^2+2xy = x(3x+2y)$

$f_y = x^2-2y-4$

To find the critical points we want $\nabla f=(f_x,f_y)=0$ so

(1) $x(3x+2y)=0$

(2) $x^2-2y-4=0$

(1) implie either $x=0$ or $3x+2y=0$

Case 1: If $x=0$ then (2) implies

$$-2y-4=0$$
$$\Rightarrow y=-2$$

Case 2: If $x\neq 0$ then $3x+2y=0$, thus

$$2y=-3x$$

Now if we plug $2y=-3x$ into (2) we get

$$x^2+3x-4=0$$
$$\Rightarrow (x+4)(x-1)=0$$
$$\Rightarrow x=-4,\ x=1$$

Now $x=-4 \Rightarrow y=\frac{-3(-4)}{2}=6$, $x=1\Rightarrow y=\frac{-3(1)}{2}=\frac{-3}{2}$

So we have there are 3 critical points given by:

$$(0,-2),\ (-4,6),\ (1,-\tfrac{3}{2})$$

Now it remains to classify them. We need to compute the hessian.

$$f_{xx} = \frac{\partial}{\partial x} f_x \qquad\qquad f_{yy} = \frac{\partial}{\partial y} f_y$$

$$= \frac{\partial}{\partial x}(3x^2+2xy) \qquad\qquad = \frac{\partial}{\partial y}(x^2-2y-4)$$

$$= 6x+2y \qquad\qquad = -2$$

Now $f_{xy} = f_{yx}$ by Clairot's theorem, so

$$f_{xy} = \frac{\partial}{\partial y} f_x$$

$$= \frac{\partial}{\partial y}(3x^2+2xy)$$

$$= 2x$$

So $D(x,y) = f_{xx} f_{yy} - f_{xy} f_{yx}$

$$= (6x+2y)(-2) - (2x)(2x)$$

$$= -4x^2 - 12x - 4y$$

- $\left.\begin{array}{l} D(0,-2) = -4(0)^2 - 12(0) - 4(-2) = 8 > 0 \\ f_{xx}(0,-2) = 6(0) + 2(-2) = -4 < 0 \end{array}\right\} \Rightarrow (0,-2)$ is a local max
- $D(-4,6) = -4(-4)^2 - 12(-4) - 4(6) = -40 < 0 \Rightarrow (-4,6)$ is a saddle
- $D(1,-\frac{3}{2}) = -4(1)^2 - 12(1) - 4(-\frac{3}{2}) = -10 < 0 \Rightarrow (1,-\frac{3}{2})$ is a saddle

Note: $f(0,-2) = 4$, $f(-4,6) = -64$, $f(1,-\frac{3}{2}) = \frac{13}{4} = 3.25$

Remarks:

① A critical point may not always exist, even though $Df=0$.

eg: $f(x,y) = \frac{y-x}{x-y} + (x-1)(y-1)$, $D(f) = \{(x,y) \mid x \neq y\}$

Let us compute ∇f. $\forall (x,y) \in D(f)$,

$$f_x = \frac{\partial}{\partial x}\left(\frac{y-x}{x-y} + (x-1)(y-1)\right) \qquad f_y = \frac{\partial}{\partial y}\left(\frac{y-x}{x-y} + (x-1)(y-1)\right)$$

$$= \frac{\partial}{\partial x}(-1 + xy - x - y + 1) \qquad = \frac{\partial}{\partial y}(-1 + xy - x - y + 1)$$

$$= y-1 \qquad = x-1$$

$$\nabla f = 0 \implies y-1=0, \; x-1=0$$

$$\implies x=1, \; y=1$$

But $(1,1) \notin D(f)$.

② Critical points tell us whether or not there is a local maximum or minimum ie they tell us around the critical point a function is optimized, but they may not the optimal value the function can take on overall.

For example: $f(x,y) = x^2+y^2$, we showed that f has a local minimum at $(0,0)$, but it can get arbitrarily large.

The question now becomes, when can be find the "optimal" value of f and how. That is what we will focus on for the rest of the section.

As always we begin with a definition:

Definition: A point (a,b) in the domain of f (ie $D(f)$) is called an absolute maximum (minimum) or global maximum (minimum) of f in the domain of f if for all $(x,y) \in D(f)$,

$$f(x,y) \leq f(a,b)$$
$$\left(f(x,y) \geq f(a,b) \right)$$

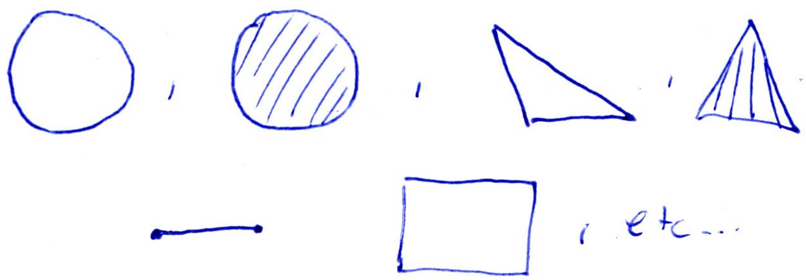
Note the difference between local maximum's and absolute maximum. If (a,b) is a local maximum then $f(a,b)$ is the largest value f can take on around (a,b), but it is possible for f to be larger further away of (a,b). Also note

local maximum $\leq$ absolute maximum.

Similar statement is true for local/global minimum.

Definition: A subset C of $\mathbb{R}^2$ is called closed and bounded or Compact If C is a "finite" region of $\mathbb{R}^2$ that contains it's own boundary/edge.

Examples:
- a circle / disc, containing it's edge
- a square, triangle, semi-circles, containing their edge
- Intervals containing end points.



, etc...

So why do we care about compact sets? Well because of the theorem below:

Theorem: A continous function $f(x,y)$ always has a global maximum/global minimum on a compact (ie. closed and bounded) region.

The above theorem is called the "extreme value theorem". You are not resposible of the statement applications of the theorem. It is only a justification to the fact that a solution will exist to the problems you are asked.

How to find an absolute maximum/minimum?

① Determine the value of f on all of the critical points inside the region

② Find the maximum or minimum on the boundary (including end points!)

③ Compare all the values:

Largest value: absolute max

Smallest value: absolute min

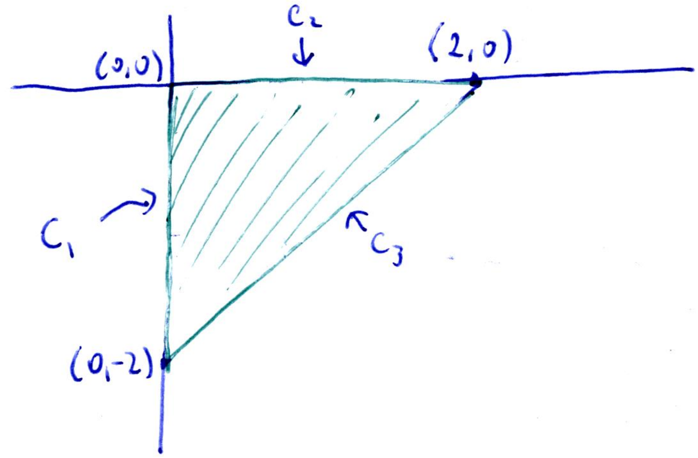
Let us do an example to illustrate this:

eg. Find the absolute max and absolute min of

$$f(x,y) = x^2+y^2-2x+2y+7$$

On the region form by the triangle $(0,0)$, $(2,0)$, $(0,-2)$.

Solution: We begin by first graphing the region.



① Let us find the critical the critical points:

$$\left.\begin{aligned} f_x &= 2x - 2 = 0 \\ f_y &= 2y + 2 = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x &= 1 \\ y &= -1 \end{aligned}$$

So $(1,-1)$ is the only critical point and it is inside our region.

Note: $f(1,-1) = 1^2 + (-1)^2 - 2 + 2(-1) + 7$
$= 5$

② We have 3 peices of the boundary C_1, C_2, C_3 respectively.

$C_1 = \{(x,y) \mid x=0, -2 \leq y \leq 0\}$

$C_2 = \{(x,y) \mid 0 \leq x \leq 2, y=0\}$

$C_3 = \{(x,y) \mid y = x-2, 0 \leq x \leq 2\}$

- On C_1 we call $g_1(y) = f(0,y)$, $-2 \leq y \leq 0$
$= y^2 + 2y + 7$

We want to find the maximum/minimum of g_1 on $-2 \leq y \leq 0$.

$g_1'(y) = 2y + 2 = 0 \Rightarrow y = -1$

So g_1 has one critical point and so we check $y=-1$, and the endpoints, $y = 0, -2$. So

$$g_1(0) = 7, \; g_1(-1) = 6, \; g_1(-2) = 7$$

• On C_2 we call $g_2(x) = f(x,0)$, $0 \leq x \leq 2$

$$= x^2 - 2x + 7$$

$g_2'(x) = 2x - 2 = 0 \Rightarrow x = 1$

So g_2 has one critical point and so we check $x = 1$, and the endpoints $x = 0, 2$.

$$g_2(0) = 7, \; g_2(1) = 6, \; g_3(2) = 7$$

• On C_3 we call $g_3(x) = f(x, x-2)$, $0 \leq x \leq 2$

$$= x^2 + (x-2)^2 - 2x + 2(x-2) + 7$$

$$= x^2 + x^2 - 4x + 4 - 2x + 2x - 4 + 7$$

$$= 2x^2 - 4x + 7$$

$g_3'(x) = 4x - 4 = 0 \Rightarrow x = 1$

So g_3 has one critical point and so we check $x=1$ and the endpoints, $x = 0, 2$.

$$g_3(0) = 7, \; g_3(1) = 5, \; g_3(2) = 7$$

So the absolute max of g_3 on the triangle is 7

absolute min of g_3 on the triangle is 5.

By this point we have seen how annoying optimizing a function on the boundary can be. Let us summarize what we did:

1. Broke the boundary up into peices
2. We found a way to define each peice in terms of one variable. This process is called parametrization.
3. After parametrizing we tuned the 2 variable optimization problem into a 1 variable one, and solved using calculus 1, techniques.

Step 2 is the hard part, some times it becomes very tough to write one variable in terms of another when the boundary gets more complicated. We need to figure out a better way when the boundary is more complicated.

Definition: A function $f(x,y)$ that we wish to optimize is called an objective function. The region that wish to approximate over is called the constraint.

Eg. In the previous example the objective function was:

$$f(x,y) = x^2 + y^2 - 2x + 2y + 7$$

The constraint was the triangle formed by $(0,0)$, $(2,0)$, $(0,-2)$.

Our goal is now to optimize some objective function with a constraint of the form

$$g(x,y) = 0.$$

For example, if I wanted to optimize a function over a circle of radius r centered at (x_0, y_0), then the circle satisfies

$$(x-x_0)^2 + (y-y_0)^2 = r^2$$

$$\Rightarrow (x-x_0)^2 + (y-y_0)^2 - r^2 = 0$$

So we define $g(x,y) = (x-x_0)^2 + (y-y_0)^2 - r^2$. With the constrant $g(x,y)=0$.

The following method is extremely useful to solve such constraint problems:

<u>Theorem</u>: (Method of Lagrange Multipliers).

Let f be the objective function and g be the constraint, with $\nabla g(x,y) \neq 0$. on the curve $g(x,y)=0$. To obtain the max/min of f subject to the constraint $g(x,y)=0$, do the following

① Find all the solutions to "λ" is called <u>Lagrange multiplier</u>

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 0 \end{cases}$$

② Apply the function to the values of (x,y) found in ①, and the largest/smallest of those is the max/min of f on $g=0$.

Instead of trying to explain all this technical jargon, lets just do an example.

<u>eg</u>. Optimize $f(x,y) = x+2y$ over ther circle of radius 2 centered at $(0,0)$.

<u>sol^n</u>: objective function: $f(x,y) = x+2y$

constraint: $g(x,y) = x^2+y^2-4=0$

Now lets compute ∇f, ∇g.

$$f_x = 1 \ , \quad f_y = 2 \ , \quad g_x = 2x \ , \quad g_y = 2y$$

$$\Rightarrow \quad \nabla f(x,y) = (1,2) \ , \quad \nabla g(x,y) = (2x, 2y)$$

So by the method of Lagrange multipliers:

$$\nabla f = \lambda \nabla g$$

- So we solve: $\begin{cases} 1 = \lambda(2x) & (1) \\ 2 = \lambda(2y) & (2) \\ x^2 + y^2 = 4 & (3) \end{cases}$

If $x \neq 0$, (1) tells us: $\lambda = \frac{1}{2x}$

So (2) tells us: $2 = \frac{1}{2x}(2y) \Rightarrow y = 2x$

So (3) tells us: $x^2 + (2x)^2 = 4$

$$\Rightarrow \quad 5x^2 = 4$$

$$\Rightarrow \quad x = \pm \frac{2}{\sqrt{5}} \ , \ \& \ y = 2x = \pm \frac{4}{\sqrt{5}}$$

So the 2 solutions are

$$\left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right), \left(\frac{-2}{\sqrt{5}}, \frac{-4}{\sqrt{5}}\right)$$

If $x = 0$ then (1) tells us $1 = 0$, which is not possible.

- $f\left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right) = \frac{2}{\sqrt{5}} + \frac{2 \cdot 4}{\sqrt{5}} = \frac{10}{\sqrt{5}} = 2\sqrt{5}$

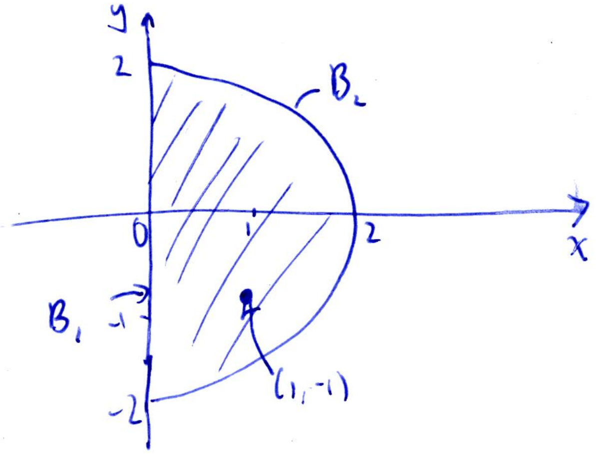
$f\left(-\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\right) = \frac{-2}{\sqrt{5}} + \frac{2(-4)}{\sqrt{5}} = \frac{-10}{5} = -2\sqrt{5}$

So the max/min of f on the circle $x^2+y^2=4$ is $2\sqrt{5}$, $-2\sqrt{5}$, respectively.

eg: Find the points on R that maximize and minimize the distance from $(1,-1)$.

$$R = \{ (x,y) \mid x^2+y^2 \le 4 \text{ and } x \ge 0 \}$$

Sol^n Let us first draw R.



First note it is equivalent to maximize/minimize the distance squared.

So the objective function is:

$$f(x,y) = (x-1)^2 + (y+1)^2$$

the constraint is :

$$R = \{ (x,y) \mid x^2+y^2 \le 4 \text{ and } x \ge 0 \}.$$

(1) Find the critical points:

$$f_x = 2(x-1), \quad f_y = 2(y+1)$$

and $\nabla f = 0 \Rightarrow x=1, \ y=-1$

So the only critical point is $(1,-1)$ which is in R.

• On B_1, we have $x=0$, and y ranges from -2, to 2.

So Let $g(y) = f(0,y)$, $-2 \le y \le 2$

$$= 1 + (y+1)^2$$

$$g'(y) = 2(y+1) = 0 \Rightarrow y = -1$$

So we need to check $y=-1$ and end points $y=-2,2$.

ie points to check are $(0,2)$, $(0,-2)$, $(0,-1)$

• on B_2 we use Lagrange:

Objective function: $f(x,y)$

constraint : $g(x,y) = x^2+y^2-4 = 0$

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow \begin{cases} 2(x-1) = \lambda 2x & (1) \\ 2(y+1) = \lambda 2y & (2) \\ x^2+y^2 = 4 & (3) \end{cases}$$

If $x \neq 0, y \neq 0$, we get

$$\lambda = \frac{x-1}{x} = \frac{y+1}{y}$$

$$\Rightarrow xy - y = xy + x$$

$$\Rightarrow y = -x$$

By (3) $\Rightarrow x^2 + (-x)^2 = 4$

$$\Rightarrow 2x^2 = 4$$

$$\Rightarrow x = \pm\sqrt{2}$$

Since $x \geq 0$ we have $x = \sqrt{2}$, and $y = -\sqrt{2}$,

The point to check is $(\sqrt{2}; -\sqrt{2})$.

If $x=0$, then $(3) \Rightarrow y = \pm 2$, and we get $(0,2)$, $(0,-2)$

If $y=0$, then $(3) \Rightarrow x = \pm 2$, but $x \geq 0$ so $x=2$ and $(2,0)$.

So the points to check are

(x,y)	$f(x,y)$
$(1,-1)$	0
$(0,2)$	10
$(0,-2)$	2
$(0,-1)$	1
$(\sqrt{2},-\sqrt{2})$	5
$(2,0)$	2

So the maximum distance is at $(0,2)$ with distance $\sqrt{10}$.
minimum distance is at $(1,-1)$ with distance 0.

Eg A manufacturer's production is modelled by the Cobb-Douglas utility function:

$$U(x,y) = 100\, x^{\frac{4}{5}} y^{1/5}$$

where x represents the units of labour and y represents the units of capital. Each unit of labour costs \$200 and each unit of capital costs 250. The total budget is \$50000. Find the maximum production level.

Note utility is a way to quantify how useful something is. In general we want to maximize utility.

Sol^{n} Objective function: $U(x,y) = 100\, x^{4/5} y^{1/5}$

constraint: $g(x,y) = 200x + 250y - 50000 = 0$, $x, y \geq 0$.

Apply Lagrange multipliers: $\nabla U = \lambda \nabla g$. If $x, y \neq 0$.

$$\begin{cases} 80\, x^{-1/5} y^{1/5} = 200\lambda & (1) \\ 20\, x^{4/5} y^{-4/5} = 250\lambda & (2) \\ 200x + 250y = 50000 & (3) \end{cases}$$

(1) divided by (2) we get

$$\frac{80\, x^{-1/5} y^{1/5}}{20\, x^{4/5} y^{-4/5}} = \frac{200\lambda}{250\lambda}$$

$$\Rightarrow 4\frac{y}{x} = \frac{4}{5}$$

$$\Rightarrow x = 5y$$

So (3) implies

$$200(5y) + 250y = 50000$$

$$\Rightarrow y = 40$$

$$\Rightarrow x = 5(40) = 200$$

So 200 units of labour, 40 units of capital, maximizes utility.